On Extending hyper4 and Knuth's Up-arrow Notation to the Reals

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Abstract

We present two extensions of the hyper4 operator. The first extension is continuous and piecewise differentiable with respect to the second operand, while the second is infinitely differentiable with respect to the second operand. Both extensions interpolate all finite iterates of hyper4 for integral values of the second operand and extend it continuously for real and complex values of the first operand. We conclude by providing for an inductive method to extend Knuth's up-arrow notation for all $x \geq 1$. These answer the question on whether such extensions for Knuth's up-arrow notation for non-integer x are possible.

1 Introduction

The *hyper4*, *tetration* or *hyperexponentiation* binary operator $hyper4(x, n)$ is defined using Maurer's and Rucker's notation for successive power iterates and the infinite iterate (see Knoebel [20, pp. 239-240]).

Definition 1.1 *For $x \in \mathbb{R}$ and $n \in \mathbb{N}$,*

$$^{n}x = \begin{cases} x & \text{, if } n = 1, \\ x^{\left(^{n-1}x\right)} & \text{, if } n > 1. \end{cases}$$

Definition 1.2 *Whenever the following limit exists and is finite,*

$$^{\infty}x = \lim_{n \to \infty} {}^{n}x$$

Knuth uses the *up-arrow* notation, which is extendible as it is defined recursively for $x, n \in \mathbb{N}$.

Definition 1.3

$$x \uparrow^{m} n = \begin{cases} x & \text{, if } n = 1, \\ x^{n} & \text{, if } m = 1, \\ x \uparrow^{m-1} \left[x \uparrow^{m} (n-1)\right] & \text{, otherwise.} \end{cases}$$

The name hyper4 comes from an alternate definition of the operator using the Ackermann function (see Knoebel [20, p. 247]):

$$\begin{aligned} A(1,m,n) &= m+n \\ A(k+1,m,1) &= m \\ A(k+1,m,n+1) &= A(k,m,A(k+1,m,n)) \end{aligned}$$

An elementary calculation shows,

$$\begin{aligned} A(1,m,n) &= m+n \\ A(2,m,n) &= m\cdot n \\ A(3,m,n) &= m^n = m\uparrow n \\ A(4,m,n) &= {}^n m = m\uparrow^2 n \\ A(5,m,n) &= m\uparrow^3 n \end{aligned}$$

Owing to the first argument of the Ackermann function, the names *tetration* (coined by Goodstein), *pentation*, *hexation*, *heptation*, *octation*, etc, are used for the higher order operators although this name usage is somewhat non-standard. For more details, consult [36].

The main question which has motivated this article is: Does there exist a *continuous* (and perhaps smooth) function which interpolates the values of ${}^n x$ and is not limited to integral values of n?

The above problem has been known under the name *continuous extension of the hyper4 or tetration or hyperexponentiation operator*, *continuous extension of the Ackermann function*, *continuous iteration of functions* and under various different names.

For a brief history please consult Knoebel [20, p. 247], Bromer [8, p. 172], Bennet [7, p. 75] and Appleby [2]. Briefly, suppose one wants to iterate $f(x)$ continuously, where $f^{(k)}$ denotes the k-th iterate of f. Abel proposed solving the problem by *linearizing* a function by addition. The problem reduces to finding a function ϕ, which satisfies $\phi(f(x)) = \phi(x)+1$. Then, $f^{(k)}(x) = \phi^{-1}(k+\phi(x))$. Schröder proposed linearizing a function by multiplication, i.e. find a function ϕ which satisfies $\phi(f(x)) = c\phi(x)$, and then $f^{(k)}(x) = \phi^{-1}(c^k\phi(x))$. For our case, $f = exp$, so one may try solving the functional equations $\phi(e^x) = \phi(x)+1$ or $\phi(e^x) = c\phi(x)$. Although Abel's equation can sometimes be solved for certain f's, (see Kuczma [21] and [22]), unfortunately determining if ϕ exists in general is difficult. Readers interested in several preliminary results, may consult Baker [4], Szekeres [31], Kneser [19], Iga [18], Rusin [29], Horowitz [17, p. 42] and Walker [35, pp. 724-728] where the author constructs infinitely differentiable solutions to the equation $F(x+1) = e^{F(x)}$, Bennet [6, pp. 48-52] and [7, p. 75], Alexander [1] for an excellent introduction to continuous iteration, and Prestrud [28, p. 48] and Hadamard [16, p. 67-68], who summarize Abel's and Schröder's results dealing with possible linearizations of functions to obtain a continuous iteration. In this paper, we construct two extensions which satisfy conditionally the functional equations $F(y+1) = x^{F(y)}$ and $F(y+1) = (e^z)^{F(y)}$.

It is useful here to introduce the notion of *hyperroot* or *tetraroot*. Consider the following: $b \cdot (1/3)$ is the solution to the equation $x \cdot 3 = b$. $b^{1/3}$ is the solution to the equation $x^3 = b$. If we continue therefore in the obvious way, ${}^{1/3}b$ is the solution to the equation ${}^{3}x = b$. Assuming $x > 1$, $b > 1$, and n a positive integer, then ${}^{n}x = b$ has a unique solution for x because ${}^{n}x$ is a strictly increasing continuous function from the interval $(1, +\infty)$ onto the interval $(1, +\infty)$. We therefore can define in general, ${}^{1/n}b$ to be the unique solution to ${}^{n}x = b$.

The main difficulty in extending the second operand of the hyper4 operator to the rationals lies with the non-commutativity of hyperexponents. That is, although $(a^b)^c = (a^c)^b = a^{bc}$, in general ${}^{c}({}^{b}a) \neq {}^{b}({}^{c}a)$. Therefore after one has defined ${}^{1/n}x$ to be the n-th order hyperroot of the real number x, there are two "natural" ways to define ${}^{m/n}x$, either as ${}^{m}({}^{1/n}x)$ or ${}^{(1/n)}({}^{m}x)$. If there is to be any hope of consistency, those two definitions should produce identical results. Unfortunately, it is easy to see that the above definitions give in general different results, unless $m = n$. Therefore, such a path that hopes to define ${}^{r}x$ for rational r, is naturally doomed to failure.

Additional hints that doom this path can be easily gotten when one considers ${}^{m/n}x$ and ${}^{m'/n'}x$, with $m/n = m'/n'$. There are problems there too, although in some of the references suggestions are given for possible definitions similar to $\lim_{k\to\infty} {}^{(km)/(kn)}x$.

In this presentation we will examine two alternate constructions, which do not suffer from an inconsistency definition-wise and the final functions acquired via the definition have the property that they preserve continuously the behavior of all the known hyperexponentials ${}^{n}x$ for natural n.

2 Notation and Preliminary Lemmas

For the terminology *analytic function* we use the one found in Churchill /Brown in [9, p. 46].

We will be using Lambert's W complex function. Although compositions of this function appear in a disguised form in Barrow [5, p. 153], De Villiers/Robinson [33, p. 14] and Knoebel [20, p. 235], most of W's essential properties are presented in Knuth /Corless /Jeffrey [10, pp. 344-349] and [11, pp. 199-204]

We specify the coefficients of the series for successive power iterates of the *exp* function, as $a_{m,n}$.

Definition 2.1

$$ {}^{m}(e^z) = \sum_{n=0}^{\infty} a_{m,n} z^n $$

Equations with complex exponents throughout this article are always understood to use the principal branch of complex exponentiation, whenever necessary: $c^w = e^{w log(c)}$, $c \neq 0$, with *log* being the principal branch of the complex *log* function.

We extend Maurer's notation to be able to work with more general power iterates.

Definition 2.2 *For* $z \in \mathbb{C} \setminus \{x \in \mathbb{R} \colon x \leq 0\}$ *and* $n \in \mathbb{N}$*,*

$$^{n}(z, w) = \begin{cases} z^{w} & \text{, if } n = 1, \\ z^{\left(^{n-1}(z,w)\right)} & \text{, if } n > 1. \end{cases}$$

Whenever the left exponent n is missing, it will be understood to be 1. Note also that $^{n}(x, 1) = {}^{n}x$.

Definition 2.3 *For* $x \geq 0$*,*

$$\lfloor x \rfloor = Int(x)$$
$$\{x\} = x - \lfloor x \rfloor$$

Differences between $\{x\}$ the fractional part and $\{x\}$ the set, will be clear from context.

Lemma 2.4 *If* $\langle r_k \rangle_{k \in \mathbb{N}}$*, is a Cauchy sequence of rationals with* $\lim_{k \to +\infty} r_k = y \in \mathbb{R}^{+} \backslash \mathbb{N}$*,* $\langle \lfloor r_k \rfloor \rangle_{k \in \mathbb{N}}$ *is also a Cauchy sequence of rationals with* $\lim_{k \to +\infty} \lfloor r_k \rfloor = \lfloor y \rfloor$*.*

Proof: Given any $\epsilon > 0$, there exists $k_0 \in \mathbb{N}$, such that for all $k > k_0$, $|r_k - y| < \epsilon$. In particular pick a k_0, that works for $\epsilon = min\{|y - [y]|, |[y] + 1 - y|\}$. It is clear then that for all $k > k_0$: $\lfloor r_k \rfloor = \lfloor y \rfloor$, therefore $|\lfloor r_k \rfloor - \lfloor y \rfloor| = 0 < \epsilon$ and the result follows. □

One may wonder here why we excluded the naturals from the above Lemma. If $\langle r_k \rangle_{n \in \mathbb{N}}$ is *any* Cauchy sequence converging to $n \in \mathbb{N}$, the Lemma may fail if we approach n from the left, since then $|\lfloor r_k \rfloor - n| = |n - 1 - n| = 1 > \epsilon$, for all k. So one should be careful to pick a Cauchy sequence that approaches n from the right.

Corollary 2.5 *If* $\langle r_k \rangle_{k \in \mathbb{N}}$ *is a Cauchy sequence of rationals with* $lim_{k \to +\infty} r_k = y \in \mathbb{R}^{+} \backslash \mathbb{N}$*, then* $\langle \{r_k\} \rangle_{k \in \mathbb{N}}$ *is also a Cauchy sequence with* $lim_{k \to +\infty} \{r_k\} = \{y\}$*.*

Proof: Write $\{r_k\} = r_k - \lfloor r_k \rfloor$ and note that the sum of two fundamental sequences is again fundamental. □

3 A Continuous Extension of hyper4

Definition 3.1 *For* $r \in \mathbb{Q}^{+}$*, define,*

$$^{r}x = \begin{cases} ^{1}(x, r) = x^{r} & \text{, if } \lfloor r \rfloor = 0, \\ ^{\lfloor r \rfloor}(x, x^{\{r\}}) & \text{, if } \lfloor r \rfloor > 0 \end{cases}$$

We can now naturally extend the definition unambiguously.

Definition 3.2 *Let $\langle r_k \rangle_{k\in\mathbb{N}}$ be a Cauchy sequence of rational numbers, with* $\lim_{k\to+\infty} r_k = y \geq 0$. *Then*

$$ {}^{y}x = \begin{cases} \lim\limits_{k\to+\infty} {}^{1}(x, r_k) = \lim\limits_{k\to+\infty} x^{r_k} & , \text{ if } \lfloor y \rfloor = 0, \\ \lim\limits_{k\to+\infty} {}^{\lfloor r_k \rfloor}(x, x^{\{r_k\}}) & , \text{ if } \lfloor y \rfloor > 0 \end{cases} $$

(For the actual details on extending the fundamental exponentials: $G_a(x) = a^x$ and $H_a(x) = x^a$ continuously over to the Reals, the reader may consult Feferman [13, p. 285]).

Note that if we fix x and we call ${}^{y}x = F(y)$, then this function satisfies the functional equation $F(y+1) = x^{F(y)}$, since $F(y+1) = {}^{\lfloor y+1 \rfloor}(x, x^{\{y+1\}}) = {}^{\lfloor y+1 \rfloor}(x, x^{\{y\}}) = x^{{}^{\lfloor y \rfloor}(x, x^{\{y\}})} = x^{F(y)}$.

Lemma 3.3 *Given $x > 0$, ${}^{n}(x, y)$ is continuous for all $y \geq 0$, and $n \in \mathbb{N}$.*

Proof: By induction on n. $x^y = {}^{1}(x, y) = e^{y log(x)}$ is continuous for all $y \geq 0$. ${}^{k+1}(x, y) = x^{{}^{k}(x,y)}$, the last being continuous as the composition of x^y, (which is continuous from the $n = 1$ step) and ${}^{k}(x, y)$ which is continuous from the inductive step, $n = k$. □

Fixing $x > 0$, it is clear that when y is away from integral values (i.e. if $y \in \mathbb{R}^+ \setminus \mathbb{N}$), we don't have a problem, since there, $\{y\} = y - \lfloor y \rfloor$ is continuous, therefore small changes in y will intuitively result in small changes for ${}^{y}x$, no matter how high the tower is. The only "suspicious" points where continuity may fail, are the natural numbers. (Because $\{y\}$ is discontinuous there). The natural numbers are *the points of transition* where the tower acquires additional exponents.

Lemma 3.4 *Given $x > 0$, ${}^{y}x$ is continuous on $y \in \mathbb{N}$.*

Proof: By induction on n. For $y = 1$, definition 3.1 gives, $\left|{}^{1+dy}x - {}^{1}x\right| = \left|{}^{\lfloor 1+dy \rfloor}(x, x^{\{1+dy\}}) - {}^{1}(x, 1)\right| = \left|{}^{1}(x, x^{dy}) - {}^{1}(x, 1)\right| = \left|x^{x^{dy}} - x^{x^{0}}\right| < \epsilon$, since by Lemma 3.3, x^{x^y} is continuous at 0.

On the other hand, $\left|{}^{1-dy}x - {}^{1}x\right| = \left|x^{1-dy} - x^{1}\right| < \epsilon$, since by Lemma 3.3, x^y is continuous at 1.

Assume now ${}^{y}x$ is continuous at $y = k$. Then $\left|{}^{k+1+dy}x - {}^{k+1}x\right| = \left|{}^{\lfloor k+1+dy \rfloor}(x, x^{\{k+1+dy\}}) - {}^{k+1}(x, 1)\right| = \left|{}^{k+1}(x, x^{dy}) - {}^{k+1}(x, 1)\right|$. The last expression is $\left|{}^{k}(x, x^{x^{dy}}) - {}^{k}(x, x^{x^{0}})\right| < \epsilon$, by the inductive step, composition and Lemma 3.3.

From the left, $\left|{}^{k+1-dx}x - {}^{k+1}x\right| = \left|{}^{\lfloor k+1-dy \rfloor}(x, x^{\{k+1-dy\}}) - {}^{k+1}(x, 1)\right| = \left|{}^{k}(x, y^{1-dy}, k) - {}^{k+1}(x, 1)\right| = \left|{}^{k}(x, x^{1-dy}) - {}^{k}(x, x^{1})\right| < \epsilon$, by the inductive step, composition and Lemma 3.3, and the Lemma follows. □

Lemma 3.4 along with continuity of ${}^{y}x$ at the non-transitional points, i.e. at $\mathbb{R}^+ \setminus \mathbb{N}$ (which follows trivially from Lemma 3.3) shows immediately the following Lemma.

Lemma 3.5 *For fixed $x > 0$, the function ${}^{y}x$ is continuous for all $y \geq 0$.*

On the other hand we have also,

Lemma 3.6 *For fixed $y > 0$, ${}^{y}x$ is continuous for all $x \in \mathbb{R}$.*

Proof: This amounts to showing that for fixed y, the function ${}^{\lfloor y \rfloor}(x, x^{\{y\}})$ is continuous. But when y is fixed, then so is $\lfloor y \rfloor = n$ and then so is $q = \{y\}$, and in this case, ${}^{\lfloor y \rfloor}(x, x^{\{y\}}) = x^{\cdot^{\cdot^{\cdot^{x^q}}}}$ ($n+1$ x's). That the last function is continuous follows easily by using induction on n, the fact that x^y is continuous for fixed y and from composition of continuous functions, with a similar argument of that in the proof of Lemma 3.3. (See also Feferman as per above). □

Knoebel [20, p. 240], Mitchelmore [26, p. 645], Ogilvy [27, p. 556] and Galidakis in [15, p. 767] establish that an algebraic infinite exponential converges if and only if its base belongs to the interval $\left[e^{-e}, e^{e^{-1}}\right] \doteq [0.06598, 1.44466]$, while Ash [3, pp. 207-208] and Macdonnell [24, pp. 301-303] establish that for $k \in \mathbb{N}$, $\lim_{c\to 0^+} {}^{2k}c = 1$ and $\lim_{c\to 0^+} {}^{2k+1}c = 0$. Whenever $c \in (0, e^{-e})$, $\langle {}^{n}c \rangle_{n\in\mathbb{N}}$ is a 2-cycle, by considering the even and odd subsequences, ${}^{2n}c$ and ${}^{2n+1}c$. The bifurcation which occurs and its behavior and properties are analyzed in Ash [3, p. 207], De Villiers/Robinson [33, p. 15] and Macdonnell [24, p. 299]. We note that the two branches stemming from the bifurcation point $\left\{e^{-e}, e^{-1}\right\}$ can be parametrized as $a^{a/(1-a)}$ and $a^{1/(1-a)}$ for appropriate positive a. (see for example Knoebel [20, p. 237] or Voles [34, p. 212]). In this case as shown in Spivak [30, p. 434], Knoebel [20, pp. 241-243], De Villiers/Robinson [33, p. 13] and Lense [23, p. 501], the two separate limits $a = \lim_{n\to\infty} {}^{2n+1}c$ and $b = \lim_{n\to\infty} {}^{2n}c$ satisfy $0 < a < h(c) < b < 1$ and the *second auxiliary equation system*,

$$\left\{\begin{matrix} a & = & c^{c^a} \\ b & = & c^a \end{matrix}\right\} \tag{3.1}$$

An analytic solution to system (3.1) is presented in a forthcoming article [14], in which we solve the n-th auxiliary equation $f^{(}p)(z) = z$ using a function similar to Lambert's W function.

At this point it is useful to look at how ${}^{y}x$ (with $x > 0$, fixed and $y > 0$) behaves relative to the results quoted above.

Figure 1 displays the behavior of the function ${}^{y}x$ at $x = e^{-e}$ which is the leftmost point of convergence of the Infinite Exponential $\langle {}^{n}x \rangle_{n\in\mathbb{N}}$. The Infinite Exponential there converges to e^{-1}.

Figure 2 displays the behavior of ${}^{y}x$ at $x = 0.02534$, where it is known that the Infinite Exponential $\langle {}^{n}x \rangle_{n\in\mathbb{N}}$ is a 2-cycle.

Figure 3 displays the behavior of the function ${}^{y}x$ at $x = e^{e^{-1}}$ which is the rightmost point of convergence of the Infinite Exponential $\langle {}^{n}x \rangle_{n\in\mathbb{N}}$. The Infinite Exponential there converges to e.

Figure 4 is a continuous interpolation between the graphs of the functions $\langle {}^{n}x \rangle_{n\in\mathbb{N}}$ for integral n using the function ${}^{y}x$. The reader may want to compare with the graphs in Ash [3], Macdonell [24, p. 299], Bromer [8, p. 170], and Voles [34].

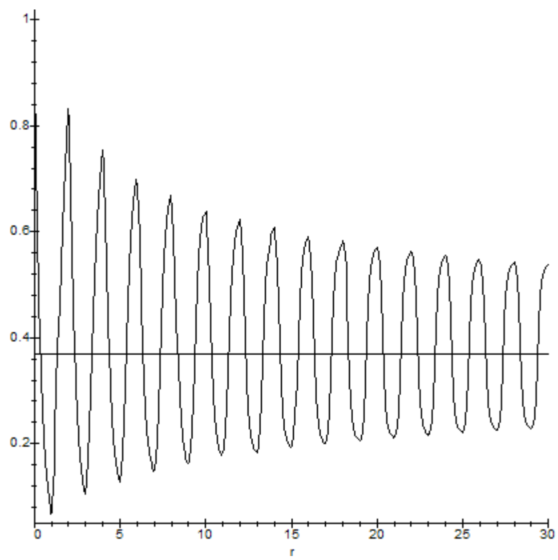

Figure 1: ${}^{y}(e^{-e}), y \in (0, 30)$ and $y = e^{-1}$

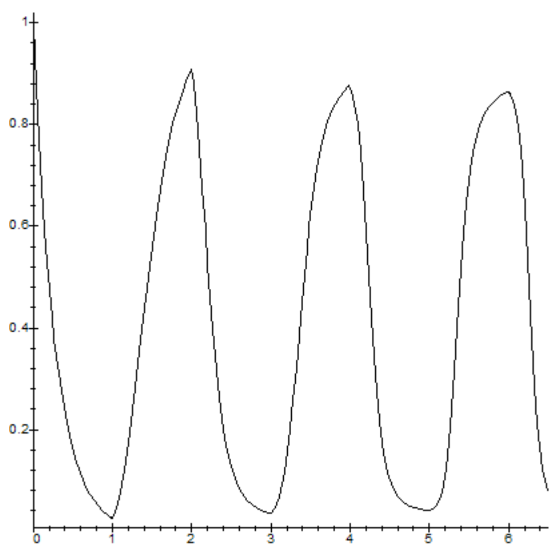

Figure 2: ${}^{y}0.02534, y \in (0, 6.5)$

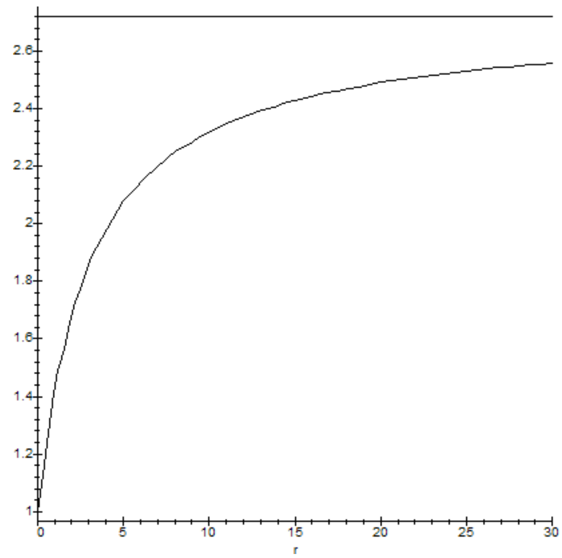

Figure 3: ${}^{y}(e^{e^{-1}}), y \in (0, 30)$ and $y = e$

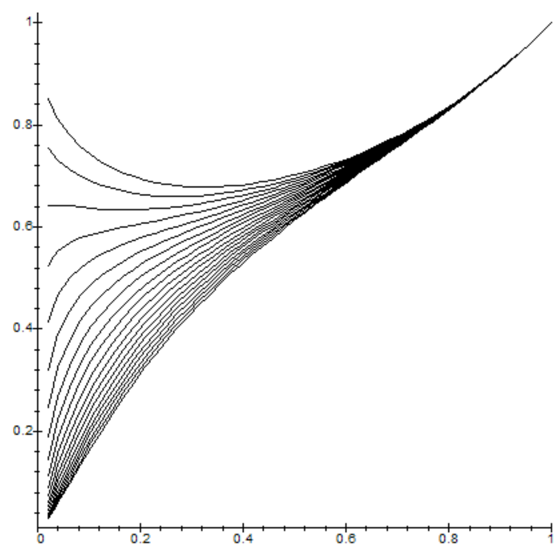

Figure 4: ${}^{2+n/20}x, n \in \{1, 2, \ldots 20\}, x \in (0, 1]$

4 Afterthoughts

What does the function ${}^{y}x$ look like for real y, intuitively? Roughly speaking, it is a growing tower of x's, with y controlling the tower's acquisition of further exponentials. The highest exponent $q = \{y\}$, always varies in $[0, 1)$. What's interesting is what happens as y approaches a natural number n from the left. When this happens, the tower still has $\lfloor y \rfloor + 1 = n - 1 + 1 = n$ exponentials. I.e. it is $x^{\cdot^{\cdot^{\cdot^{x^q}}}}$, n x's. $(q = 1^-)$.

When y passes over n becoming an integer and then growing further, the top exponential x^q stabilizes to x^1 and a new exponential starts on top of it, with a new q very close to 0. I.e. $x^{\cdot^{\cdot^{\cdot^{x^q}}}}$, $n + 1$ x's. $(q = 0^+)$.

The *exp* function (which is what's working under the scenes) makes the transition seamless, resulting in a smooth transitioning into the exponential tower that has one more exponential at its top.

Intuitively, one could perhaps visualize this function as an already infinite exponential, in which successive *state exponents* get activated continuously. I.e. $(x^{y_1})^{(x^{y_2})^{\cdots (x^{y_n})^{\cdots}}}$. The state exponents y_i of the function are determined uniquely by the decomposition of each real y as,

$$\begin{aligned} y &= \lfloor y \rfloor + \{y\} = n + q,\ n \in \mathbb{N},\ q \in [0, 1), \\ y_i &= 1 \text{ for } i <= n, \\ y_{n+1} &= q \\ y_j &= 0,\ j > n + 1. \end{aligned} \tag{4.1}$$

y_i always ranges in $[0, 1)$ and the function starts with $y_i = 0$, for all $i > 1$. As y in ${}^{y}x$ moves in $(0, +\infty)$, it causes (by virtue of its own unique decomposition (4.1)) an activation of those exponents y_i, according to the scheme above, causing successive acquisitions of higher exponentials, which force the tower to grow indefinitely, preserving however continuously the functions ${}^{n}x$ as y passes through the naturals. And that's what we wanted.

Note that there is no problem if x is complex if we use the principal branch of the complex *log* function. In other words, ${}^{y}z$, $y \geq 0$, $z \in \mathbb{C}$, works as expected. In fact, tracing the behavior of ${}^{y}z$ can reveal a plethora of information about the convergence of various iterated exponentials. In particular, Macintyre in [25] argues as follows. "If we note that the function $w = exp(\pi i z/2)$ maps the half strip $0 < \Re(z) < 1$, $\Im(z) > 0$ on to the quadrant $|w| < 1$, $0 < arg(w) < \pi/2$, the convergence (of $i^{i^{\cdot^{\cdot^{\cdot}}}}$) is proved by considering iterations of this mapping..." and then hand draws the attracting basin. Continuously tracing ${}^{y}i$, for $y \geq 0$, produces a surprisingly similar result to the image of the attracting basin shown in [25]. The result is shown in figure 5.

On the other hand tracing ${}^{y}(-1)$ for example, can reveal an infinity of new surprises. Even though ${}^{n}(-1) = -1$, for all $n \in \mathbb{N}$, the values of ${}^{y}(-1)$ for $y \in \mathbb{R}^+ \setminus \mathbb{N}$ are literally quite complex.

The only shortcoming of the above extension is that it is only piecewise differentiable. In particular, its derivative is not continuous at the Naturals. To

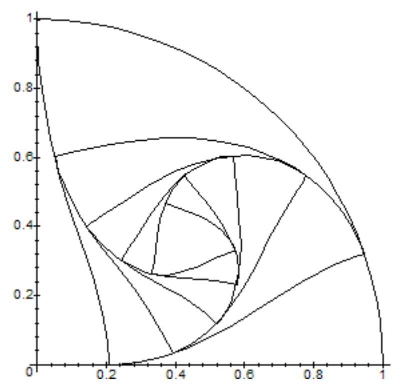

Figure 5: ${}^{y}i, y \in [0, 15]$

see this, note that if the derivative at some natural number was continuous, the two limits $\lim_{y \to 1^-} dy/dy$ and $\lim_{y \to 0^+} d(x^y)/dy$ should be equal, however the first limit equals 1 while the second equals $xln(x)$.

5 An Infinitely Differentiable Extension of hyper4

In [15, p. 776] we show Corollary 5.1 and Lemma 5.2:

Corollary 5.1 *For $m \in \mathbb{N}$, ${}^{m}(e^z)$ is entire, with series expansion*

$$ {}^{m}(e^z) = \sum_{n=0}^{m} \frac{(n+1)^n}{(n+1)!} z^n + \sum_{n=m+1}^{\infty} a_{m,n} z^n $$

where $a_{m,n}$ are given by the recursion:

$$ a_{m,n} = \begin{cases} 1 & , \textit{if } n = 0, \\ \frac{1}{n!} & , \textit{if } m = 1, \\ \frac{\sum_{j=1}^{n} j a_{m,n-j} a_{m-1,j-1}}{n} & , \textit{otherwise} \end{cases} \tag{5.1} $$

Lemma 5.2 *If $m, n \in \mathbb{N}$ and $m \geq n$, then $a_{m,n} = a_{n,n}$*

Using the relation ${}^{k}n = log_n({}^{k+1}n)$ (which follows from the definition of hyper4), one defines ${}^{1}n = log_n({}^{2}n) = log_n(n^n) = n$, therefore ${}^{0}n = log_n({}^{1}n) =$

m:n	0	1	2	3	4	5	6
0	1	0	0	0	0	0	0
1	1	1	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{24}$	$\frac{1}{120}$	$\frac{1}{720}$
2	1	1	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{41}{24}$	$\frac{49}{30}$	$\frac{1057}{720}$
3	1	1	$\frac{3}{2}$	$\frac{8}{3}$	$\frac{101}{24}$	$\frac{63}{10}$	$\frac{6607}{720}$
4	1	1	$\frac{3}{2}$	$\frac{8}{3}$	$\frac{125}{24}$	$\frac{49}{5}$	$\frac{12847}{720}$
5	1	1	$\frac{3}{2}$	$\frac{8}{3}$	$\frac{125}{24}$	$\frac{54}{5}$	$\frac{16087}{720}$
6	1	1	$\frac{3}{2}$	$\frac{8}{3}$	$\frac{125}{24}$	$\frac{54}{5}$	$\frac{16807}{720}$

Table 1: $a_{m,n}$ for ${}^{m}(e^z)$, $(m,n) \in \{0,1,...,6\} \times \{0,1,...,6\}$

1. Accordingly we augment recursion (5.1) as follows:

$$a_{m,n} = \begin{cases} 1 & \text{, if } m = n = 0, \\ 0 & \text{, if } m = 0 \text{ and } n \neq 0, \\ \frac{1}{n!} & \text{, if } m = 1, \\ \frac{\sum_{j=1}^{n} j a_{m,n-j} a_{m-1,j-1}}{n} & \text{, otherwise} \end{cases} \tag{5.2}$$

Table 1 gives all the coefficients based on recursion (5.2) up to $m = n = 6$.

We will use a variant of a well known C^∞ function which can be found in Vasy [32], Farassat [12, p. 3] and elsewhere.

$$f(x) = \begin{cases} e^{-\frac{1}{x}} & \text{, if } x > 0, \\ 0 & \text{, if } x \leq 0 \end{cases} \tag{5.3}$$

Elementary calculus shows that $f(1-x^2)$ is symmetric about the origin and there, $f(1-x^2)$ attains its maximum, $f(1) = e^{-1}$, while $f(1-x^2) = 0$ for $|x| \geq 1$. We first change the support of f to be the interval $(-1/2, 1/2)$, so we consider the function $f(1/4 - x^2)$.

Definition 5.3

$$\phi(x) = f(1/4 - x^2) = \begin{cases} e^{\frac{4}{4x^2-1}} & \text{, } \textit{if } |x| < 1/2, \\ 0 & \text{, } \textit{otherwise} \end{cases}$$

$\phi(x)$ is shown in figure 6. Consider the functions $\phi(x - (m - 1/2))$. Set $\int_{m-1}^{m} \phi(t - m - 1/2)dt = A_m$. Since $\phi(x - (m - 1/2))$ is a right translation of $\phi(x)$, it follows that for all $m, n \in \mathbb{N}$, $A_m = A_n$. We first normalize ϕ with respect to its integral, so we set $\chi_m(x) = (a_{m,n} - a_{m-1,n})\phi(x - (m - 1/2))/A_m$, and finally,

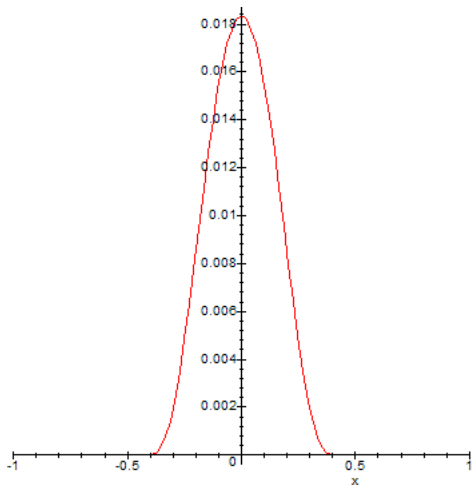

Figure 6: $\phi(x)$

Definition 5.4 *For all $m \in \mathbb{N}$, $n \in \mathbb{N} \cup \{0\}$, $x \geq 0$ and with initial values for $a_{m,n}$ as in recursion* (5.2)*,*

$$\alpha_n(x) = \begin{cases} 1 & , \text{ if } n = 0 \\ \int\limits_0^x \sum\limits_{m=1}^n \chi_m(t)dt & , \text{ if } n \neq 0, \end{cases}$$

If $x = m > 0$ then since χ_m is zero outside its support $(m - 1/2, m + 1/2)$, $\alpha_n(m) = \sum_{k=1}^m \int_{k-1}^k \chi_k(t)dt = \sum_{k=1}^m (a_{k,n} - a_{k-1,n}) \int_{k-1}^k \phi(t - (m - 1/2))dt/A_m = a_{m,n} - a_{0,n} = a_{m,n}$. It is also clear that if $x \geq n$ then $\alpha_n(x) = a_{n,n}$. Figure 7 is the graph of $\alpha_3(x)$.

Lemma 5.5 *If $x \geq 0$, $\alpha_n(x)$ is infinitely differentiable with respect to x.*

Proof: If $x \geq n > 0$, $\alpha_n(x) = a_{n,n} =$ constant, so derivatives of all orders exist and are 0. If $x < n$, then $d^p \left[\alpha_n(x)\right]/dx^p = d^{p-1}\left[\sum_{m=1}^n \chi_m(x)\right]/dx^{p-1} = \sum_{m=1}^n d^{p-1}\left[\chi_m(x)\right]/dx^{p-1}$, since the sum is finite, and $\chi_m(x)$ is a right translation of $\phi(x)$ for which derivatives of all orders exist everywhere and the Lemma follows. $\square$

We are ready for the extension:

Definition 5.6 *With $\alpha_n(x)$ as in definition 5.4 and $y \geq 0$,*

$$^{y}(e^z) = \sum_{n=0}^{\infty} \alpha_n(y) z^n$$

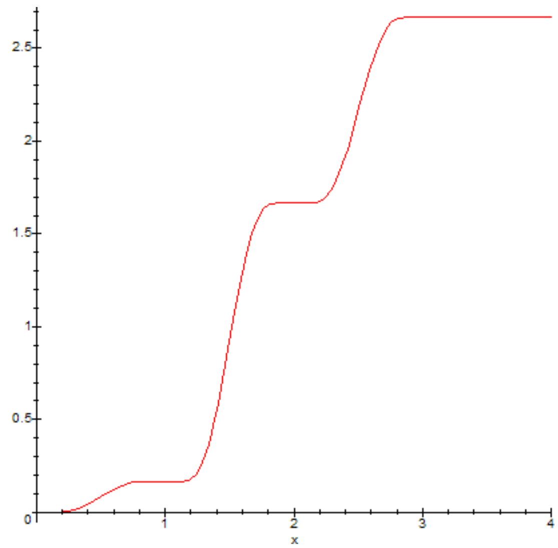

Figure 7: $\alpha_3(x)$, $x \in [0, 4]$

Note that if we fix z and call $F(m) = {}^{m}(e^z)$, then F satisfies the functional equation $F(m+1) = (e^z)^{F(m)}$, for $m \in \mathbb{N}$. $\alpha_n(m+1) = a_{m+1,n} = \left(\sum_{j=1}^{n} j a_{m+1,n-j} a_{m,j-1}\right)/n = \left(\sum_{j=1}^{n} j \alpha_{n-j}(m+1)\alpha_{j-1}(m)\right)/n$, consequently, $F(m+1) = (e^z)^{F(m)}$ by Corollary 5.1.

We first prove convergence.

Lemma 5.7 *If $y \geq 0$, then $S_k(z) = \sum_{n=0}^{k} \alpha_n(y) z^n$ converges uniformly on compact subsets of $\mathbb{C}$.*

Proof: If $y = 0$ then $S_k(z) = 1 \to 1$. Fix $y > 0$ and $z \in U \subset \mathbb{C}$, U compact. Then, $m - 1 \leq y < m$, for some $m \in \mathbb{N}$, therefore $|\alpha_n(y)| = \left|\alpha_n(m-1) + \int_{m-1}^{y} \chi_m(t)dt\right| \leq \left|\alpha_n(m-1) + \int_{m-1}^{m} \chi_m(t)dt\right| = |\alpha_n(m)| = a_{m,n}$, for all $n \in \mathbb{N}$, hence for all $z \in U$ and each $y > 0$, $|\alpha_n(y)z^n| \leq a_{m,n} |z|^n = M_n$ and $\sum_{n=0}^{\infty} M_n = {}^{m}(e^{|z|})$ by Corollary 5.1, so by the Weierstrass M-test, the series $S_k(z)$ converges (absolutely and) uniformly on compact subsets and the Lemma follows. $\square$

If $y = 0$, then ${}^{y}(e^z) = 1$ as required by the definition of hyper4, while if $y = m \in \mathbb{N}$, then ${}^{y}(e^z)$ coincides with the corresponding expansion for the tower ${}^{m}(e^z)$ in Corollary 5.1. Therefore ${}^{y}(e^z)$ interpolates all finite towers of iterates of e^z. The important question now is if this interpolation is not only continuous, but C^∞ with respect to y. We are ready for the second result of this paper.

Lemma 5.8 *If $y \geq 0$, then ${}^{y}(e^z)$ is infinitely differentiable with respect to y.*

Proof: Since $S_k(z)$ converges uniformly on compact subsets, we can differentiate term by term.

$$\begin{aligned}\frac{d^p}{dy^p}\left[{}^y(e^z)\right] &= \frac{d^p}{dy^p}\left[\sum_{n=0}^{\infty}\alpha_n(y)z^n\right]\\ &= \sum_{n=0}^{\infty}\frac{d^p}{dy^p}\left[\alpha_n(y)\right]z^n\end{aligned}$$

$d^p\left[\alpha_n(y)\right]/dy^p$ exists in the domain of α_n, for all $p \geq 1$ by Lemma 5.5, therefore $d^p\left[{}^y(e^z)\right]/dy^p$ also exists for $y \geq 0$ for all $p \geq 1$ and the Lemma follows. □

We can now define a corresponding C^∞ function that interpolates between all the finite power iterates of z.

Definition 5.9 *With $\alpha_n(y)$ as in definition 5.3, $y \geq 0$ and z in the domain of the principal branch of log,*

$${}^y z = \sum_{n=0}^{\infty}\alpha_n(y)log(z)^n$$

6 Afterthoughts

In [15] we show that as m grows larger, the coefficients given by (5.1) eventually stabilize to the coefficients of the expansion for $W(-z)/(-z)$, where W is Lambert's W function. There, we show the following Corollary.

Corollary 6.1 *${}^\infty(e^z)$ is analytic at the origin, with series expansion:*

$${}^\infty(e^z) = \sum_{n=0}^{\infty}\frac{(n+1)^n}{(n+1)!}z^n$$

and radius of convergence: $R_s = e^{-1}$.

It is interesting to check the limit of ${}^y(e^z)$ as y grows without bound.

$$\begin{aligned}\lim_{y\to+\infty}{}^y(e^z) &= \lim_{y\to+\infty}\sum_{n=0}^{\infty}\alpha_n(y)z^n\\ &= \sum_{n=0}^{\infty}\lim_{y\to+\infty}\alpha_n(y)z^n\\ &= \sum_{n=0}^{\infty}a_{n,n}z^n\\ &= \sum_{n=0}^{\infty}\frac{(n+1)^n}{(n+1)!}z^n\end{aligned}$$

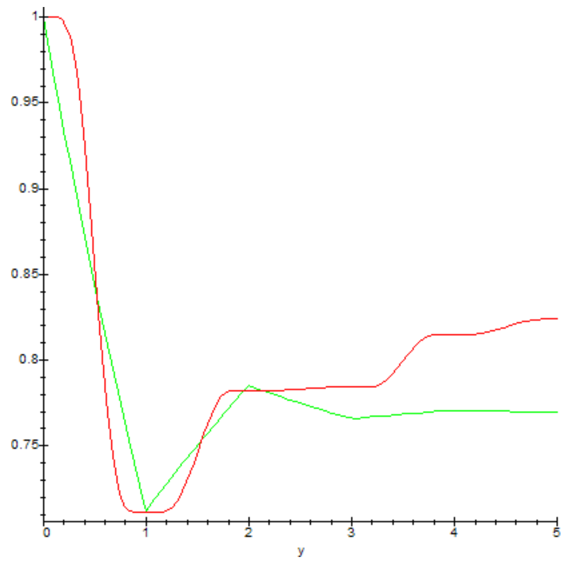

Figure 8: ${}^{y}(e^{-0.34})$, $x \in [0, 5]$

using Lemma 5.7, Definition 5.4 and Corollaries 5.1 and 6.1. The last series converges for $|z| \leq e^{-1}$ and equals the expansion of $W(-z)/(-z)$ for such z, which agrees with the fact that ${}^{\infty}(e^{z}) = W(-z)/(-z)$ as shown in [15, p. 775]. (In general, we know that $\lim_{n \to +\infty} {}^{n}(e^{x})$ exists for $-e \leq x \leq e^{-1}$. See for example Knoebel [20, p. 240], Mitchelmore [26, p. 645], Ogilvy [27, p. 556] and [15, p. 775]).

Because the radius of convergence of the above series is e^{-1}, the behavior of any numerical computations for the extension ${}^{y}(e^{z})$ that depend on the coefficients (5.2) for convergence, will most likely be bad past this radius of convergence. Indeed, we can set up some Maple code to calculate ${}^{y}(e^{x})$ for $|x| \leq e^{-1}$, but past these bounds (or disk in the complex case) any code will likely produce garbage. And this is to be expected. If convergence of the series expansions of the iterates ${}^{m}(e^{x})$ was everywhere nice, the limit function would be entire, but it's not, although all finite iterates are entire.

In general, calculating ${}^{y}(e^{z})$ (resp. ${}^{y}z$) even when $|z| \leq e^{-1}$ (resp. when $|log(z)| \leq e^{-1}$), is quite difficult, because it depends on explicitly defining $\alpha_n(y)$. The α_n functions are all defined piecewise and as n grows larger, they become harder and harder to define explicitly. A numerical approximation for ${}^{y}(e^{-0.34})$ using just 6 terms in the extension ($n = 5$), coupled with the extension of section 3, is shown in figure 8, with $y \in [0, 5]$, where it is known that $lim_{y \to +\infty} {}^{y}(e^{-0.34}) = W(0.34)/0.34 \doteq 0.76973$

7 Extending Knuth's Up-Arrow Notation

The two extensions presented in sections 3 and 5, show that an infinity of similar extensions is possible. It is possible for example to construct an extension as in Section 3, with the function ${}^{y}z$ being linear or quadratic in $[0,1]$ and still satisfying the functional equation $F(y+1) = x^{F(y)}$. Similarly, by changing the support of the ϕ functions in Section 5, one could conceivably construct an infinite family of infinitely differentiable functions ${}^{y}(e^{z})$ which still interpolate all the finite iterates of e^{z}. One then extends the three argument Ackermann function $a(4,x,y)$ as $a(4,x,y) = {}^{y}x$, using any of these extensions.

We now temporarily turn back using Knuth's up-arrow notation from definition 1.3. The problem with this notation seems that it can only be used with $x \in \mathbb{N}$. However, upon closer inspection it can now be extended inductively to include all $x \geq 1$, as follows. We have already defined $x \uparrow^2 y$ as $\sum_{n=0}^{\infty} \alpha_n(y) log(x)^n$, therefore we ask if $x \uparrow^2 x = \sum_{n=0}^{\infty} a_n(x) log(x)^n$ makes sense. If we restrict x as $x \geq 1$, then $log(x) \geq 0$, consequently $log(x)^n \geq 0$, and therefore $x \uparrow^2 x \geq 0$, since $\alpha_n(x) \geq 0$. Once we have defined $x \uparrow^2 x$, we can define $x \uparrow^3 m$.

Definition 7.1 *For $x \geq 1$, and $m \in \mathbb{N}$,*

$$x \uparrow^3 m = \sum_{n=0}^{\infty} \alpha_n(x \uparrow^3 (m-1)) log(x)^n$$

Defining $x \uparrow^4 m$, requires that we define $x \uparrow^3 x$, which is done in a similar way as in section 5 or 3. For example, we start by writing the series we have defined so far and rename the coefficients into $b_{m,n}$.

$$\begin{aligned}
x \uparrow^3 1 &= \sum_{n=0}^{\infty} \frac{1}{n!} log(x)^n = \sum_{n=0}^{\infty} b_{1,n} log(x)^n = x \\
x \uparrow^3 2 &= \sum_{n=0}^{\infty} \alpha_n(x) log(x)^n = \sum_{n=0}^{\infty} b_{2,n} log(x)^n \\
x \uparrow^3 3 &= \sum_{n=0}^{\infty} \alpha_n(x \uparrow^2 x) log(x)^n = \sum_{n=0}^{\infty} b_{3,n} log(x)^n \\
&\dots \\
x \uparrow^3 m &= \sum_{n=0}^{\infty} \alpha_n(x \uparrow^3 (m-1)) log(x)^n = \sum_{n=0}^{\infty} b_{m,n} log(x)^n
\end{aligned} \tag{7.1}$$

At this point it would help if we knew how the sequence $x \uparrow^3 m$ behaves for fixed $x > 1$, so we momentarily turn our attention to this sequence. For this we will need three Lemmas:

Lemma 7.2 *With $\alpha_n(x)$ defined as in 5.4, $\alpha_n(x)$ is a nondecreasing function of x.*

Proof: If $x_1 < x_2$ then $\alpha_n(x_1) \leq \alpha_n(x_1) + \int_{x_1}^{x_2} \chi_m(t)dt = \alpha_n(x_2)$ and the Lemma follows. □

Corollary 7.3 *With $x \uparrow^2 y$ defined as in 5.9, for fixed $x > 1$, the function $x \uparrow^2 y$ is a nondecreasing function of $y > 1$.*

Proof: $\alpha_n(y)$ is a nondecreasing function of $y \geq 1$ by Lemma 7.2, so if $y_1 < y_2$ then $x \uparrow^2 y_1 \leq x \uparrow^2 y_2$ using definition 5.4. □

Lemma 7.4 *With $x \uparrow^2 y$ defined as in 5.9, for fixed $x > 1$, the sequence $\langle x \uparrow^3 m \rangle_{m \in \mathbb{N}}$ is a nondecreasing sequence on m.*

Proof: We have to show that $x \uparrow^3 (m+1) \geq x \uparrow^3 m$. We use induction on m. For $m = 1$, we show that $x \uparrow^3 2 \geq x \uparrow^3 1$ or equivalently using definition 1.3, $x \uparrow^2 x = {}^x x \geq x$. Note that if $x > 1$ then $x = e^{log(x)} = \sum_{n=0}^{\infty} log(x)^n/n!$ while $x \uparrow^2 x = \sum_{n=0}^{\infty} \alpha_n(x) log(x)^n$, using definition 5.9. $\alpha_n(x)$ is nondecreasing, using Lemma 7.2, so if $x > 1$ then $\alpha_n(x) \geq \alpha_n(1) = a_{1,n} = 1/n!$ using recursion (5.1), so this part follows immediately. Assume the assertion holds for $m = k$, so $x \uparrow^3 (k+1) \geq x \uparrow^3 k$ holds. It now follows that $x \uparrow^2 (x \uparrow^3 (k+1)) \geq x \uparrow^2 (x \uparrow^3 k)$, using Corollary 7.3, consequently $x \uparrow^3 (k+2) \geq x \uparrow^3 (k+1)$ and the Lemma follows. □

Lemma 7.5 *The sequence $\langle x \uparrow^3 m \rangle_{m \in \mathbb{N}}$ is bounded on $(1, e^{e^{-1}}]$.*

Proof: If $x \in (1, e^{e^{-1}}]$, the sequence $\langle x \uparrow^2 n \rangle_{n \in \mathbb{N}}$ is bounded above by e, (See Knoebel [20, p. 240], Mitchelmore [26, p. 645], Ogilvy [27, p. 556] and [15, p. 775]), so we use induction. For $m = 2$, $x < e^{e^{-1}} < 2$ so $x \uparrow^2 x \leq x \uparrow^2 2$, since $x \uparrow^2 y$ is nondecreasing by Corollary 7.3, consequently $x \uparrow^3 2 \leq x \uparrow^2 2 \leq e$. Assume that $x \uparrow^3 k \leq e$. Then $x \uparrow^3 (k+1) = x \uparrow^2 (x \uparrow^3 k) \leq x \uparrow^2 e \leq x \uparrow^2 3 \leq e$ using corollary 7.3 again twice and the Lemma follows. □

Lemmas 7.4 and 7.5 imply,

Corollary 7.6 *If $x \in (1, e^{e^{-1}}]$, then the sequence $\langle x \uparrow^3 m \rangle_{m \in \mathbb{N}}$ converges.*

It is interesting to note here that when the sequence $\langle x, {}^x x, {}^{{}^x x} x, \ldots \rangle = \langle x \uparrow^3 m \rangle_{m \in \mathbb{N}}$ converges, the limit l must satisfy ${}^l x = l$, in a similar way as with the exponential tower ${}^n x$, where the limit then satisfies $x^l = l$. (For the latter, see Barrow [5, p. 153]). In other words, l has to be a fixed point of the function $f(y) = {}^y x = x \uparrow^2 y$. But according to the definition of such exponents in the introduction, this is equivalent to writing $x = {}^{1/y} y$. In other words, whenever $1 \leq x \leq e^{e^{-1}}$, the function $x = {}^{1/y} y$ is a partial inverse of the infinite hyperexponential ${}^{{}^{\cdot^{\cdot^{\cdot}}} x} x$, in a similar way $x = y^{1/y}$ is a partial inverse of the infinite exponential $x^{x^{\cdot^{\cdot^{\cdot}}}}$. (For the latter see Knoebel [20, pp. 239,247]).

Lemma 7.7 *For each $n \in \mathbb{N}$, $\alpha_n(x)$ is uniformly continuous on $[0, +\infty)$*

Proof: $\alpha_n(x)$ is continuous on $[0, n]$ therefore it is uniformly continuous there. On the other hand, $\alpha_n(x)$ is constant (and equals $a_{n,n}$) on $[n, +\infty)$, and the Lemma follows. $\square$

Now we show that the coefficients $b_{m,n}$ eventually stabilize as in Lemma 5.2.

Lemma 7.8 *For all $\epsilon > 0$ and each $n \in \mathbb{N}$, there exists $l \in \mathbb{R}$ and $m_0 \in \mathbb{N}$, such that if $m \geq m_0$ then $|b_{m,n} - \alpha_n(l)| < \epsilon$.*

Proof: The sequence $\langle x \uparrow^3 m \rangle_{m\in\mathbb{N}}$ is positive and nondecreasing by Lemma 7.4, so it either diverges or it converges to a positive number $l \in \mathbb{R}$. If it diverges, then for each $n \in \mathbb{N}$, there exists a m_0, such that for all $m \geq m_0$, $x \uparrow^3 m \geq n$, therefore for all $m \geq m_0$, $\alpha_n(x \uparrow^3 m) = \alpha_n(n) = a_{n,n}$, by definition 5.4. If on the other hand it converges, then clearly $\langle \alpha_n(x \uparrow^3 m) \rangle_{m\in\mathbb{N}}$ converges to $\alpha_n(l)$, using Lemma 7.7 and the Lemma follows. $\square$

Since the $b_{m,n}$ eventually stabilize, we can continue the process. We again first normalize ϕ with respect to its integral, so we set $\psi_m(x) = (b_{m,n} - b_{m-1,n})\phi(x - (m - 1/2))/A_m$, and then we interpolate between the $b_{m,n}$ as with definition 5.4.

Definition 7.9 *For all $m \in \mathbb{N}$, $n \in \mathbb{N} \cup \{0\}$, $x \geq 0$ and with initial values for $b_{m,n}$ as in equations* (7.1)*,*

$$\beta_n(x) = \begin{cases} 1 & \text{, if } n = 0 \\ \int\limits_0^x \sum\limits_{m=1}^{\infty} \psi_m(t)dt & \text{, if } n \neq 0, \end{cases}$$

If $x = m > 0$ then since ψ_m is zero outside its support $(m - 1/2, m + 1/2)$, $\beta_n(m) = \sum_{k=1}^{m} \int_{k-1}^{k} \psi_k(t)dt = \sum_{k=1}^{m}(b_{k,n} - b_{k-1,n}) \int_{k-1}^{k} \phi(t - (m-1/2))dt/A_m = b_{m,n} - b_{0,n} = b_{m,n}$.

When $x \in (1, e^{e^{-1}}]$, the sequence $\langle x \uparrow^3 m \rangle_{m\in\mathbb{N}}$ converges by Corollary 7.6 , so the coefficients stabilize "asymptotically" to $\alpha_n(l)$, while when $x > e^{e^{-1}}$, the sequence diverges and the coefficients $b_{m,n}$ stabilize as constants $\alpha_n(n) = a_{n,n}$. Accordingly in the later case, the upper bound of the summation in definition 7.9 becomes finite, since then $b_{m,n} - b_{m-1,n} = a_{n,n} - a_{n,n} = 0$, consequently $\psi_m(x) = (b_{m,n} - b_{m-1,n})\phi(x - (m-1/2))/A_m$ vanishes identically, for all $m-1 \geq n$.

$\beta_n(x)$ is again infinitely differentiable, as in Lemma 5.5 so we continue.

Definition 7.10 *With $\beta_n(x)$ as in definition 7.9 and $y \geq 0$,*

$$(e^z) \uparrow^3 y = \sum_{n=0}^{\infty} \beta_n(y) z^n$$

Definition 7.11 *With $\beta_n(y)$ as in definition 7.9, $y \geq 0$ and x in the domain of the principal branch of log,*

$$x \uparrow^3 y = \sum_{n=0}^{\infty} \beta_n(y) log(x)^n$$

Once we have defined $x \uparrow^3 y$, we can define $x \uparrow^4 m$,

Definition 7.12 *For $x \geq 1$, and $m \in \mathbb{N}$,*

$$x \uparrow^4 m = \sum_{n=0}^{\infty} \beta_n(x \uparrow^4 (m-1)) log(x)^n$$

Now we can now write down the series for $x \uparrow^4 m$ as in (7.1).

$$\begin{aligned}
x \uparrow^4 1 &= \sum_{n=0}^{\infty} \frac{1}{n!} log(x)^n = \sum_{n=0}^{\infty} c_{1,n} log(x)^n = x \\
x \uparrow^4 2 &= \sum_{n=0}^{\infty} \beta_n(x) log(x)^n = \sum_{n=0}^{\infty} c_{2,n} log(x)^n \\
x \uparrow^4 3 &= \sum_{n=0}^{\infty} \beta_n(x \uparrow^3 x) log(x)^n = \sum_{n=0}^{\infty} c_{3,n} log(x)^n \\
&\dots \\
x \uparrow^4 m &= \sum_{n=0}^{\infty} \beta_n(x \uparrow^4 (m-1)) log(x)^n = \sum_{n=0}^{\infty} c_{m,n} log(x)^n
\end{aligned} \tag{7.2}$$

The process continues inductively in an exactly similar way as with $x \uparrow^3 m$.

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